Further Analyses of Moon's Errors with Mean Elongation as argument, 1847–1901. By P. H. Cowell.

The present analyses are based upon the errors after thirteen corrections have been applied to Hansen's tabular places. These corrections are given in previous papers, where the periods of analysis are also defined.

In the annexed table the first column gives the number of the period of analysis. The next two columns are copied from last month's paper. The next three columns are fresh matter, and they are sufficiently distinguished by their headings. The meaning of the word "apparent" is that these are the corrections that are found by analysis upon the successive assumptions that each one alone exists. The subsequent columns, headed "resolved values," take account of the co-existence of the various separate errors. The relation between the "apparent" and "resolved" values is clearly dependent upon the distribution of the observations relatively to the age of the Moon. Relations were investigated, based upon the whole of the 48 periods of analysis (1847–1901). These relations should not strictly be applied to the individual periods, for any particular period may have an abnormal distribution of observations, but the error due to this cause has been allowed to coalesce with the accidental errors in the belief that they will not be so distributed as to sensibly affect the coefficients of deduced periodic corrections.

I now investigate the formulæ used. The notation is μ, δ_1, δ_2, Δ_1, Δ_2 stand respectively for the correction to semi-diameter, coefficients of sin D, sin 2D, cos D, cos 2D (resolved values). Accented letters denote apparent values. It will be seen that μ, δ_1, δ_2 fall into one group, Δ_1, Δ_2 into a second group, and that the two groups are kept separate.

First as to μ, δ_1, δ_2; in the December *Monthly Notices* I obtained from 5,647 observations normal equations:

$$5647\mu + 3788\delta_1 - 2327\delta_2 = \Sigma\pm\epsilon,$$
$$3788\mu + 3074\delta_1 - 1366\delta_2 = \Sigma\epsilon \,.\, \sin \mathrm{D},$$
$$-2327\mu - 1366\delta_1 + 2645\delta_2 = \Sigma\epsilon \,.\, \sin 2\mathrm{D},$$

ϵ being the error of any observation and the 48 times 400 lunar days being taken together.

For each of the present periods of analysis I put, n denoting the number of observations.

$$\mu' = \frac{1}{n}\Sigma\pm\epsilon$$

$$\delta'_1 = \frac{2}{n}\Sigma\epsilon \sin \mathrm{D}.$$

$$\delta'_2 = \frac{5647}{2645n}\Sigma\epsilon \sin 2\mathrm{D}.$$

Analyses of Tabular minus *Observed Longitudes of Moon for Periods* 86–133 *in Tenths of a Second of Arc.*

Period.	Apparent Values of: Semi-Diameter Term.	Co-efficient of sin D.	Co-efficient of sin 2D.	Co-efficient of cos D.	Co-efficient of cos 2D.	Resolved Values of: Semi-Diameter Term.	Co-efficient of sin D.	Co-efficient of sin 2D.	Co-efficient of cos D.	Co-efficient of cos 2D.
86	+ 1	+ 2	0	− 5	+ 2	− 1	+ 3	+ 1	− 8	− 2
87	− 8	− 11	+ 1	− 2	+ 1	− 13	+ 2	− 10	− 2	0
88	− 2	− 1	+ 3	− 10	+ 4	− 7	+ 9	+ 1	− 16	− 4
89	0	0	0	− 6	+ 4	0	0	0	− 3	+ 3
90	− 4	− 5	+ 3	− 7	+ 4	− 6	+ 2	− 1	− 6	+ 1
91	+ 3	+ 5	− 2	0	+ 2	0	+ 5	0	+ 8	+ 6
92	− 5	− 8	+ 6	− 3	+ 2	+ 2	− 8	+ 3	− 2	+ 1
93	− 2	0	+ 5	− 5	+ 4	− 9	+ 13	+ 4	0	+ 5
94	− 3	− 5	0	+ 1	− 1	− 2	− 4	− 4	− 1	− 1
95	− 6	− 9	+ 4	+ 7	− 4	− 4	− 4	− 1	+ 6	− 1
96	− 8	− 12	+ 5	− 1	− 4	− 5	− 6	− 3	− 19	− 15
97	− 11	− 13	+ 11	+ 5	− 2	− 16	+ 9	+ 1	+ 8	+ 2
98	− 6	− 9	+ 5	+ 1	+ 2	− 2	− 5	0	+ 11	+ 8
99	0	− 1	+ 1	− 4	+ 2	+ 5	− 6	+ 2	− 5	− 1
100	+ 3	+ 4	− 5	− 3	+ 2	+ 1	+ 1	− 4	− 2	+ 1
101	0	− 1	0	+ 1	− 1	+ 4	− 6	+ 1	− 1	− 1
102	+ 3	+ 4	− 5	+ 10	− 7	+ 1	+ 1	− 4	+ 3	− 5
103	+ 3	+ 4	− 4	+ 2	+ 1	+ 2	+ 1	− 2	+ 10	+ 6
104	− 3	− 6	+ 4	0	+ 1	+ 6	− 12	+ 3	+ 4	+ 3
105	+ 1	+ 1	+ 2	− 4	+ 2	+ 5	− 4	+ 5	− 5	− 1
106	+ 1	+ 3	0	+ 3	− 1	− 5	+ 9	0	+ 6	+ 2
107	+ 1	+ 2	0	+ 5	− 2	− 1	+ 3	+ 1	+ 8	+ 2
108	+ 4	+ 6	0	+ 3	− 1	+ 5	+ 1	+ 5	+ 6	+ 2
109	+ 8	+ 13	− 4	0	0	+ 2	+ 10	+ 3	0	0
110	+ 3	+ 2	− 3	− 1	0	+ 11	− 11	+ 1	− 3	− 2
111	− 2	− 3	0	− 3	+ 2	− 3	− 1	− 3	− 2	+ 1
112	0	− 1	0	− 5	+ 4	+ 4	− 6	+ 1	0	+ 5
113	+ 1	+ 2	− 3	+ 2	+ 1	− 4	+ 5	− 4	+ 10	+ 6
114	+ 2	+ 3	− 4	− 12	+ 6	− 2	+ 3	− 4	− 13	− 1
115	+ 3	+ 3	0	− 9	+ 6	+ 10	− 7	+ 5	− 4	+ 4
116	+ 3	+ 3	− 4	− 14	+ 6	+ 6	− 5	− 2	− 22	− 5
117	+ 4	+ 4	− 5	− 5	+ 3	+ 8	− 7	− 1	− 4	+ 2
118	+ 3	+ 4	− 2	− 2	+ 1	+ 4	0	+ 1	− 2	0
119	+ 1	+ 2	− 4	+ 5	− 3	− 5	+ 5	− 6	+ 4	− 2
120	0	0	− 3	− 4	0	− 3	+ 2	− 5	− 13	− 7
121	+ 3	+ 5	0	0	+ 1	+ 2	+ 4	+ 4	+ 4	+ 3
122	+ 8	+ 12	− 5	− 3	+ 1	+ 5	+ 6	+ 3	− 6	− 2
123	+ 3	+ 5	− 3	− 2	+ 1	− 1	+ 6	− 1	− 2	0
124	+ 3	+ 5	− 2	− 6	+ 4	0	+ 5	0	− 3	+ 3
125	+ 1	+ 1	− 6	− 6	+ 4	− 3	+ 1	− 8	− 3	+ 3
126	− 4	− 6	+ 2	+ 1	− 1	− 3	− 3	− 2	− 1	− 1
127	+ 2	+ 2	− 2	− 4	+ 4	+ 4	− 4	0	+ 3	+ 6
128	0	− 1	0	0	− 1	+ 4	− 6	+ 1	− 4	− 3
129	− 4	− 4	+ 4	− 4	+ 2	− 9	+ 7	0	− 5	− 1
130	− 3	− 5	+ 2	− 5	+ 1	0	− 5	0	− 12	− 5
131	− 5	− 6	+ 4	− 4	+ 2	− 8	+ 4	− 1	− 5	− 1
132	− 1	− 2	0	− 3	+ 2	+ 1	− 3	− 1	− 2	+ 1
133	− 1	− 2	+ 3	+ 4	+ 1	+ 4	− 5	+ 4	+ 17	+ 10

Subsequently I deduce μ, δ_1 and δ_2 from the formulæ

$$\begin{aligned} 5647\mu+3788\delta_1-2327\delta_2 &= 5647\mu' \\ 3788\mu+3074\delta_1-1366\delta_2 &= \tfrac{1}{2}5647\delta_1' \\ -1366\mu+2327\delta_1+2645\delta_2 &= 2645\delta_2' \end{aligned}$$

It will be seen that μ', δ_1', δ_2' are purely arbitrary auxiliary quantities. The treatment is unfortunately not quite symmetrical. I wish that I had taken for δ_1', $\frac{1}{2}\cdot\frac{5647}{3074}$ times the value which I have taken. The explanation is that when I came to Δ_1, Δ_2 I was forced into definitions corresponding to that of δ_2'; I was in time to define δ_2' accordingly, but too late for δ_1'. Of course nothing is lost but symmetry.

Solving the equations for μ, δ_1, δ_2, in terms of μ', δ'_1, δ'_2, I obtain

$$\begin{aligned} \mu &= 7\cdot285\mu'-3\cdot980\delta'_1+1\cdot079\delta'_2 \\ \delta_1 &= -7\cdot960\mu'+5\cdot537\delta'_1-0\cdot597\delta'_2 \\ \delta_2 &= +2\cdot306\mu'-0\cdot637\delta'_1+1\cdot646\delta'_2 \end{aligned}$$

and with the help of these equations the resolved values have been obtained from the apparent values.

It follows that

$$n\mu = \Sigma\epsilon\{\pm 7\cdot285-7\cdot960\sin D+2\cdot306\sin 2D\}$$

$$\frac{n}{2}\cdot\delta_1 = \Sigma\epsilon\{\mp 3\cdot980+5\cdot537\sin D-0\cdot637\sin 2D\}$$

$$\frac{n}{2}\delta_2 = \Sigma\epsilon\{\pm 1\cdot153-0\cdot637\sin D+1\cdot756\sin 2D\}$$

Throughout any period of analysis the value of D is supposed to increase by $720°\div 57$ between one transit of the Moon and the next, and the small error of this assumption is prevented from accumulating by introducing the necessary discontinuity in passing from one period of analysis to the next. Only 57 values of D are therefore recognised—namely, multiples of $720°\div 57$; and the above factors merely change sign as we pass from the multiple by p to the multiple by $57-p$.

I have therefore tabulated for values of D

$$=\frac{720°}{57}p\ (p=3, 4 \ldots . 25)$$

the factors for $n\mu$, $\frac{n}{2}\delta_1$ and $\frac{n}{2}\delta_2$ respectively, viz.:

$$\begin{aligned} &\pm 7\cdot285-7\cdot960\sin D+2\cdot306\sin 2D \\ &\mp 3\cdot980+5\cdot537\sin D-0\cdot637\sin 2D \\ &\pm 1\cdot153-0\cdot637\sin D+1\cdot756\sin 2D \end{aligned}$$

p	Factor for $n\mu$.	Factor for $\frac{n\delta_1}{2}$	Factor for $\frac{n\delta_2}{2}$	p	Factor for $n\mu$.	Factor for $\frac{n\delta_1}{2}$	Factor for $\frac{n\delta_2}{2}$
3	+4·7	−1·2	+2·5	15	−5·3	+2·9	−0·5
4	+3·4	−0·3	+2·4	16	−2·6	+1·4	+0·3
5	+2·1	+0·4	+2·0	17	−0·6	+0·2	+0·9
6	+0·8	+1·1	+1·4	18	+0·9	−0·8	+1·1
7	−0·5	+1·5	+0·6	19	+1·7	−1·4	+0·9
8	−1·4	+1·7	−0·1	20	+1·6	−1·6	+0·5
9	−1·8	+1·6	−0·7	21	+1·0	−1·7	−0·2
10	−1·4	+1·1	−1·0	22	0·0	−1·3	−1·0
11	−0·3	+0·3	−1·0	23	−1·3	−0·8	−1·7
12	+1·5	−0·8	−0·6	24	−2·7	−0·1	−2·2
13	+3·9	−2·1	+0·1	25	−4·0	+0·7	−2·5
14	+6·6	−3·6	+0·9				

For full Moon $p = 14\frac{1}{4}$.

Secondly as to $\Delta_1\ \Delta_2$; I put

$$\epsilon = \Delta_1(\cos D - c_1) + \Delta_2(\cos 2D - c_2)$$

when $c_1\ c_2$ are the average values of cos D, cos 2D respectively.

The normal equations are then

$$\Delta_1\Sigma(\cos D - c_1)^2 + \Delta_2\Sigma(\cos D - c_1)(\cos 2D - c_2) = \Sigma\epsilon(\cos D - c_1)$$
$$\Delta_1\Sigma(\cos D - c_1)(\cos 2D - c_2) + \Delta_2\Sigma(\cos 2D - c_2)^2 = \Sigma\epsilon(\cos 2D - c_2)$$

The coefficients on the left-hand side are

$$\Sigma(\cos D - c_1)^2 = \Sigma(\cos^2 D - c_1^2) = \frac{n}{2}\{1 + c_2 - 2c_1^2\}$$

$$\Sigma(\cos D - c_1)(\cos 2D - c_2) = \Sigma(\cos D_1 \cos 2D - c_1c_2)$$
$$= \frac{n}{2}\{c_1 + c_3 - 2c_1c_2\}$$

$$\Sigma(\cos 2D - c_2)^2 = \Sigma(\cos^2 2D - c_2^2) = \frac{n}{2}\{1 + c_4 - 2c_2^2\}$$

where c_3, c_4 denote the mean values of cos 3D, cos 4D respectively.

My paper in 1903 December enables us to evaluate these quantities as

$$\frac{n}{2}\{1 - 0\cdot09 - 0\cdot46\} = \frac{n}{2}\cdot 0\cdot45 = \frac{n}{2}a$$

$$\frac{n}{2}\{-0\cdot48 - 0\cdot09\} = -\frac{n}{2}\ 0\cdot57 = \frac{n}{2}h$$

$$\frac{n}{2}\{1 + 0\cdot06 - 0\cdot01\} = \frac{n}{2}\ 1\cdot05 = \frac{n}{2}b$$

We shall subsequently require

$$ab = 0{\cdot}47,\ ah = -0{\cdot}26,\ bh = -0{\cdot}60,\ h^2 = 0{\cdot}32,\ ab - h^2 = 0{\cdot}148$$

The normal equations are therefore:

$$\left(\frac{n}{2}\Delta_1\right) + \frac{h}{a}\left(\frac{n}{2}\Delta_2\right) = \Sigma\epsilon\left(\frac{\cos D - c_1}{a}\right) = \frac{n}{2}\Delta'_1$$

$$\frac{h}{b}\left(\frac{n}{2}\Delta_2\right) + \left(\frac{n}{2}\Delta_2\right) = \Sigma\epsilon\left(\frac{\cos 2D - c_2}{b}\right) = \frac{n}{2}\Delta'_2$$

The apparent values Δ'_1, Δ'_2, which may be viewed as arbitrary auxiliary quantities, are defined as above; and their values have been actually calculated from the formula

$$\Delta'_1 = \frac{2}{n}\Sigma\epsilon\frac{\cos D - c_1}{a} = \frac{2}{n}\Sigma\epsilon(\cos D + 0{\cdot}48) \times 2{\cdot}27$$

$$\Delta'_2 = \frac{2}{n}\Sigma\epsilon\frac{\cos 2D - c_2}{b} = \frac{2}{n}\Sigma\epsilon(\cos 2D + 0{\cdot}09) \times 0{\cdot}94$$

which correspond with sufficient accuracy to the algebraical formulæ.

Then solving for Δ_1, Δ_2 we have

$$\Delta_1 = \frac{ab}{ab-h^2}\Delta'_1 - \frac{bh}{ab-h^2}\Delta'_2 = +3{\cdot}2\Delta'_1 + 4{\cdot}1\Delta'_2$$

$$\Delta_2 = \frac{-ah}{ab-h^2}\Delta'_1 + \frac{ab}{ab-h^2}\Delta'_2 = +1{\cdot}7\Delta'_1 + 3{\cdot}2\Delta'_2$$

The following table is similar to a preceding one:

p.	Factor for $\frac{n}{2}\Delta'_1$.	Factor for $\frac{n}{2}\Delta'_2$.	Factor for $\frac{n}{2}\Delta_1$.	Factor for $\frac{n}{2}\Delta_2$.
3	+2·88	+0·32	+10·5	+5·9
4	+2·54	−0·09	+ 7·8	+4·0
5	+2·11	−0·47	+ 4·8	+2·1
6	+1·66	−0·74	+ 2·3	+0·4
7	+1·16	−0·86	+ 0·2	−0·8
8	+0·66	−0·79	− 1·1	−1·4
9	+0·18	−0·55	− 1·7	−1·4
10	−0·25	−0·20	− 1·6	−1·1
11	−0·61	+0·22	− 1·0	−0·3
12	−0·91	+0·60	− 0·4	+0·4
13	−1·09	+0·88	+ 0·1	+1·0
14	−1·18	+1·02	+ 0·4	+1·2
15	−1·16	+0·98	+ 0·3	+1·2
16	−1·02	+0·76	− 0·1	+0·7

p.	Factor for $\frac{n}{2}\Delta'_1$.	Factor for $\frac{n}{2}\Delta'_2$.	Factor for $\frac{n}{2}\Delta_1$.	Factor for $\frac{n}{2}\Delta_2$.
17	−0·77	+0·41	− 0·8	0·0
18	−0·45	0·00	− 1·4	−0·8
19	−0·05	−0·39	− 1·8	−1·3
20	+0·40	−0·69	− 1·6	−1·5
21	+0·91	−0·84	− 0·5	−1·1
22	+1·41	−0·82	+ 1·2	−0·2
23	+1·88	−0·62	+ 3·5	+1·2
24	+2·34	−0·29	+ 6·3	+3·0
25	+2·72	+0·11	+ 9·2	+5·0

The following table gives the weights of the various quantities when determined from n observations each of weight unity, and also the sum of their values for the 48 periods without regard to sign :

Quantity.	Sum of Numerical Values.	Weight.
μ'	146	n
δ'_1	208	$\frac{1}{2}n$ approximately
δ'_2	131	$\frac{1}{2}n$ approximately
Δ'_1	192	$\frac{1}{2}n\,.\,a = \frac{1}{2}n \div (1{\cdot}5)^2$
Δ'_2	113	$\frac{1}{2}n\,.\,b = \frac{1}{2}n$ approximately
μ	208	$n \times \frac{781}{5647} = n \div (2{\cdot}7)^2$
δ_1	235	$\frac{1}{2}n \times \frac{1030}{5647} = \frac{1}{2}n \div (2{\cdot}3)^2$
δ_2	117	$\frac{1}{2}n \times \frac{3228}{5647} = \frac{1}{2}n \div (1{\cdot}3)^2$
Δ_1	284	$\frac{1}{2}n\,\frac{ab-h^2}{b} = \frac{1}{2}n \div (3{\cdot}7)^2$
Δ_2	147	$\frac{1}{2}n\,\frac{ab-h^2}{a} = \frac{1}{2}n \div (1{\cdot}7)^2$

Judging from the last eight quantities alone, the probable error of a single observation is $\pm 1''{\cdot}4$. The first two quantities, however, give about double this result, a value which is clearly too large. The weighted mean is $\pm 1''{\cdot}7$ and agrees well with the estimates based upon other analyses.

Taking means for the 48 periods, I obtain

$$\begin{aligned} \mu &= -0''{\cdot}07 \\ \delta_1 &= 0{\cdot}00 \\ \delta_2 &= -0{\cdot}07 \\ \Delta_1 &= -0{\cdot}28 \\ \Delta_2 &= +0{\cdot}10 \end{aligned}$$

Of these quantities Δ_1 alone exceeds its probable error, and there is no gravitational explanation of a cos D term.

The three quantities μ, δ_1 and δ_2 are liable to probable errors of nearly $0''\cdot10$. Taking, however, the values as given above, the results of my Paper in December require slight corrections, and I now obtain

Correction to tabular semi-diameter $-0''\cdot47$

Observed Parallactic Inequality $-126''\cdot46+1''\cdot56 = -124''\cdot90$

Observed Variation $2370''\cdot14+0''\cdot07 = 2370''\cdot21$

The corrections to the December paper are,

$$\text{for } \mu - 0''\cdot12, \text{ for } \delta_1 + 0''\cdot12, \text{ for } \delta_2 - 0''\cdot04$$

and it will be seen that these numbers are approximately proportional to the factors for day 14. The values in the December Paper are probably slightly erroneous for the reason noted in *Monthly Notices*, December, p. 96, paragraph (i), but the cause there noted will only account for a small half of the discordances. The remaining part is, as far as I know, to be attributed to purely accidental differences. It must be remembered that several corrections have been applied to each observation in the interval between the two investigations. I incline, for the reasons stated, to the belief that the present values are more accurate.

The solar parallax, deduced on the assumption that there is no monthly term in the semi-diameter, has now been raised from $8''\cdot76$ by $+0''\cdot018$ by Professor Brown's calculations (see *Monthly Notices*, 1904 April) and by $+0''\cdot009$ on account of the revised result of the present paper. The value thus becomes

$$8''\cdot787 \pm 0''\cdot007.$$

Solving for $\omega-\omega'$ and ☊, I find no considerable terms except

$$+0''\cdot33 \sin ☊ \sin \text{D}.$$

This is nearly the same term, as I found a month ago, before taking account of the variation. There is no term in cos ☊ cos D, and as the coefficient $0''\cdot33$ is only about three times the probable error, I have ceased to think that the term is certainly real.

There is reason to think that the coefficients of $\sin(g-g')$ and $\sin(g-g'+2\omega-2\omega')$ in Hansen's Tables ($+148''\cdot03$ and $-28''\cdot60$) require algebraic diminution by about $0''\cdot10$ each, subject to a probable error as large as the quantity itself. Hansen's coefficients are clearly more accurate than Delaunay's.

Newcomb's corrections are now applied in the *Nautical Almanac* in a manner equivalent to

$$\text{N}\{1+\tfrac{1}{9}\cos g+\tfrac{2}{100}\cos 2\text{D}+\tfrac{2}{100}\cos(2\text{D}-g)\}$$

nearly. I have only applied the first two terms $\text{N}\{1+\frac{1}{9}\cos g\}$

to the tabular places before 1883. Immediately preceding 1883, $N = -11''$, and the two omitted terms are, for a short time,

$$-0''\cdot 2 \cos 2D - 0''\cdot 2 \cos (2D - g).$$

There seems to be a very faint trace of this in the values of Δ_2, and a more pronounced trace in the values of Δ'_2 (see periods 110–117), but I find no trace whatever in the coefficients of cos $(2D-g)$ published in a previous paper. This is not unnatural, for quantities of $0''\cdot 2$ cannot be obtained from the analysis of a very short period.

On the Comparison between the purely Theoretical and Observed Places of the Moon. By E. Nevill.

The appearance in the last number of the *Monthly Notices* (1903 November) of Mr. Cowell's Note on the Errors of the Moon's Tabular Longitude recalls the result of a more complete comparison between observation and the theory of the Moon as developed by Delaunay's method which I had intended to send to the Society six or seven years ago, but was prevented by reasons no longer existing.

In his interesting note Mr. Cowell remarks that Hansen's *Tables de la Lune* may be brought into much better accord with observation by applying corrections to bring its data into accord with the theoretical results obtained by Professor Hill and M. Radau, and especially by making use of the correction to one of the *Venus* terms of long period suggested by Professor Newcomb. But Mr. Cowell does not give the requisite figures to show clearly to what extent the discord between observation and theory is reduced by these corrections, nor does he give the data for showing distinctly what is still more important, the systematic periodical discordance remaining outstanding.

This omission suggested the abstracting and forwarding to the Society the results of the comparison between pure theory and observation yielded by my earlier investigation, which does clearly show these most important points and afford material for testing some of the conclusions stated by Mr. Cowell in his published paper.

It has been already shown that as far as the perturbations are due to the direct disturbing action of the Sun, there is no systematic difference of any importance between the theory embodied in Hansen's Tables and that which would be obtained from Delaunay's theory.

But when the supplementary terms are considered which arise from the disturbing action of the planets, the figure of the Earth, &c., then there exists grave divergency between the